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A new geometrical approach to one-parameter spatial motion

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Abstract The geometry and kinematics of one-parameter dual spherical motions is presented using Study's dual-line coordinates. The relations between invariants of the axodes are expressed in simple form with geometrical reasoning and explanation. In terms of this, the dual version of associated space curves is demonstrated for a ruled surface to be associated with the axodes of the motion. The relative motion between the axodes is used for deriving the geometry and kinematics of the paths instantaneously traced in the fixed space by a line associated with the moving axode. Especially, the distribution parameter and the inflection-line congruence are investigated. Furthermore, a new metric is developed and used to investigate the geometrical properties and kinematics of line trajectory as well as Disteli axis. Finally, as an application an example is put forward and explained in detail.

Keywords Disteli's formula · E. Study's map · Line congruence · Ruled surface

1 Introduction

Line trajectories are important in kinematic design because they can be identified with lines of kinematic elements of a particular mechanism. In spatial motion, the trajectories of oriented lines embedded in a moving rigid body are generally ruled surfaces. In kinematics, we are interested in studying the intrinsic properties of line trajectories from the concepts of a ruled surface in differential geometry. Thus, the differential geometry of a ruled surface is important in the study of rational design problems in spatial mechanisms. An important analytical tool in the study of line trajectories is the introduction of dual numbers which were first introduced by Clifford [1] and rediscovered by Study [2]. A comprehensive analysis of dual numbers and their applications to the kinematic analysis of spatial linkages was conducted by Yang [3]. Bottema and Roth [4] include a treatment of theoretical kinematics using dual numbers. Dual

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numbers are extremely useful for spatial mechanisms, since there is a vast literatures on this branch of classical differential line geometry and spatial mechanisms; for more details see [3–17]

In spatial kinematics, the geometry of infinitesimally separated positions of a system of rigid bodies is a difficult but important and interesting subject that has been widely studied during the past decades. It may lead to very complex computations in the most compact form and searches for their most rational organization. This target motivates a great deal of research on the fundamental operations and the algebraic structures lying behind kinematics methods. There exists a vast literature on the subject including several monographs, for example [18–23].

It is well known in spatial kinematics that the instantaneous screw axis (ISA) at a prescribed instant can be determined by the first-order derivatives of a conjugate motion of one degree of freedom, i.e., the translational velocity and the angular velocity at an assigned point of a moving body (see, for example, [4]). It is also well known that the aggregation of the ISAs of all instants from a pair of ruled surfaces, called the moving and fixed axodes, with ISA as their generating line in the moving space and in the fixed space, respectively. Since the fixed and moving centroids from a logical starting point on which to build a geometric theory of the instantaneous planar kinematics of a moving body, as pointed out by Hunt [24], we may also expect the axodes to occupy a similar important fundamental place in the spatial kinematic geometry of a rigid body. These are expressible in terms comparable to those of the plane, but they have not appeared so far.

The motion of one member of a mechanism of rigid members, be it produced by cams, gears, or linkages, is best expressed relative to some second member. In the resulting two-body problem it is convenient to fix the latter and to describe the motion relative to a fixed frame of reference fixed in it. This paper provides a new geometric and kinematic approach to one-parameter spatial motion for the calculation of instantanous invariants based on information specifying the motion of the axodes. In terms of this, for one-parameter spatial motions the integral invariants of a ruled surface, generated by a fixed line in the moving space, are related to the instantaneous invariants of the motion. This novel approach is used to reformulate the geometrical properties of line trajectories. Then the distribution parameter and the inflection-line congruence are examined in detail, and a new proof of the well-known formulae for the Disteli axis is given. Further, a new metric is demonstrated and used to investigate the geometrical properties and the kinematics of line trajectories with the Disteli axis. Finally, a practical example is put forward and explained.

2 Elements of screw calculus

We start our discussion by reviewing some of the basic concepts of dual numbers. Dual numbers are the set of all pairs of real numbers written as

$$A = a + \varepsilon a^*,\ a,\ a^* \in \mathbb{R}, \tag{2.1}$$

where the dual unit ε satisfies the relationships

$$\varepsilon \neq 0,\ \varepsilon 1 = 1\varepsilon,\ \varepsilon^2 = 0. \tag{2.2}$$

The application of line geometry and dual-number representation of line trajectories has been developed by Blaschke [24] and Bottema and Roth [4]. A more recent description of this representation can be found in [1,3,20–24]; the dual number is used to recast the point-displacement relationship into relationships of lines.

As stated, the dual numbers were first introduced by Clifford [1] after which Study [2] used it as a tool for his research on differential-line geometry. Given the dual numbers $A = a + \varepsilon a^*$, and $B = b + \varepsilon b^*$, the rules for combination can be defined as:

$$\left.\begin{array}{l} \textit{Equality}: A = B \Longleftrightarrow a = b,\ a^* = b^*, \\ \textit{Addition}: A + B = (a + b) + \varepsilon(a^* + b^*), \\ \textit{Multiplication}: AB = ab + \varepsilon(a^*b + ab^*). \end{array}\right\} \tag{2.3}$$

The set of dual numbers, denoted as D, forms a commutative group under addition. The associative laws hold for multiplication and dual numbers are distributive. As a result, the division of dual numbers is defined as:

$$\frac{A}{B} = \frac{a}{b} + \varepsilon\left(\frac{a^*b - ab^*}{b^2}\right), \quad b \neq 0. \tag{2.4}$$

A dual number is called a pure dual when

$$A = \varepsilon a^*. \tag{2.5}$$

Division by a pure dual number is not defined. An example of a dual number is the dual angle between two skew lines in space defined as:

$$\Theta = \theta + \varepsilon\theta^*, \tag{2.6}$$

where θ is the projected angle between the lines and θ^* is the minimal distance between the lines along their common perpendicular line.

A differentiable function $f(x)$ can be defined for a dual variable $f(x + \varepsilon x^*)$ by expanding the function using a Taylor series:

$$f(x + \varepsilon x^*) = f(x) + \varepsilon x^* \frac{\mathrm{d}f(x)}{\mathrm{d}x}.$$

Thus, we have the following:

$$\left.\begin{array}{l} \sin^{-1}(\theta + \varepsilon\theta^*) = \sin^{-1}\theta + \varepsilon\frac{\theta^*}{\sqrt{1-\theta^2}}, \\ \cos^{-1}(\theta + \varepsilon\theta^*) = \cos^{-1}\theta - \varepsilon\frac{\theta^*}{\sqrt{1-\theta^2}}, \\ \tan^{-1}(\theta + \varepsilon\theta^*) = \tan^{-1}\theta + \varepsilon\theta^* \sec^2\theta. \end{array}\right\} \tag{2.7}$$

Other functions may also be defined in this manner. It may also be shown that, for a positive integer n,

$$A^n = a^n + \varepsilon n a^* a^{n-1} = a^n\left(1 + \varepsilon n\frac{a^*}{a}\right). \tag{2.8}$$

2.1 Ruled surface with dual representation

An oriented line L in the three-dimensional Euclidean space E^3 can be determined by a point $\mathbf{p} \in L$ and a normalized direction vector $\mathbf{a}$ of L, i.e., $\|\mathbf{a}\| = 1$. To obtain components for L, one forms the moment vector

$$\mathbf{a}^* = \mathbf{p} \times \mathbf{a}, \tag{2.9}$$

with respect to the origin point in E^3. If $\mathbf{p}$ is substituted by any point

$$\mathbf{q} = \mathbf{p} + \lambda\mathbf{a};\ \lambda \in \mathbb{R},$$

on L, Eq. 2.9 implies that $\mathbf{a}^*$ is independent of $\mathbf{p}$ on L. The two vectors $\mathbf{a}$ and $\mathbf{a}^*$are not independent of one another; they satisfy the following relationships:

$$\langle\mathbf{a}, \mathbf{a}\rangle = 1, \quad \langle\mathbf{a}^*, \mathbf{a}\rangle = 0. \tag{2.10}$$

The six components $a_i,\ a_i^*(i = 1, 2, 3)$ of $\mathbf{a}$, and $\mathbf{a}^*$are called the normalized Plucker coordinates of the line L. Hence the two vectors $\mathbf{a}$ and $\mathbf{a}^*$ determine the oriented line L.

Conversely, any six-tuple $a_i,\ a_i^*(i=1,2,3)$ with

$$a_1^2+a_2^2+a_3^2=1,\quad a_1a_1^*+a_2a_2^*+a_3a_3^*=0, \tag{2.11}$$

represent a line in the three-dimensional Euclidean space E^3. Thus, the set of all oriented lines in the three-dimensional Euclidean space E^3 is in one-to-one correspondence with pairs of vectors in E^3 subject to the relationships in Eq. 2.11

For vectors $(\mathbf{a}^*,\mathbf{a})\in E^3\times E^3$ we define the set

$$D^3=D\times D\times D=\{\mathbf{A}=\mathbf{a}+\varepsilon\mathbf{a}^*;\ \varepsilon\neq 0,\ \varepsilon 1=1\varepsilon,\ \varepsilon^2=0\}. \tag{2.12}$$

Then for any vectors $\mathbf{A}$, and $\mathbf{B}$ in D^3, the scalar product is defined by

$$\langle\mathbf{A},\mathbf{B}\rangle=\langle\mathbf{a},\mathbf{b}\rangle+\varepsilon(\langle\mathbf{a}^*,\mathbf{a}\rangle+\langle\mathbf{a},\mathbf{b}^*\rangle), \tag{2.13}$$

and the norm of $\mathbf{A}$ is defined by

$$\|\mathbf{A}\|=\|\mathbf{a}\|+\varepsilon\frac{\langle\mathbf{a}^*,\mathbf{a}\rangle}{\|\mathbf{a}\|},\quad \|\mathbf{a}\|\neq 0. \tag{2.14}$$

Hence, we may write the dual vector $\mathbf{A}$ as a dual multiplier of a dual vector in the form

$$\mathbf{A}=\|\mathbf{A}\|\mathbf{U}, \tag{2.15}$$

where $\mathbf{U}$ is referred to as the axis. The ratio

$$h=\frac{\langle\mathbf{a}^*,\mathbf{a}\rangle}{\|\mathbf{a}\|}, \tag{2.16}$$

is called the pitch along the axis $\mathbf{U}$. If $h=0$ and $\|\mathbf{a}\|=1$, $\mathbf{A}$ is an oriented line, and when h is finite, $\mathbf{A}$ is a proper screw; and when h is infinite, $\mathbf{A}$ is called a couple.

A dual vector with norm equal to unity is called a dual unit vector. From Eq. 2.14 it is easy to show that a dual unit vector satisfies the relationships in Eq. 2.11. Hence, each oriented line $L=(\mathbf{a},\mathbf{a}^*)\in E^3$ is represented by a dual unit vector

$$\mathbf{A}=\mathbf{a}+\varepsilon\mathbf{a}^*. \tag{2.17}$$

The dual unit sphere in D^3 is defined as follows:

$$\langle\mathbf{A},\mathbf{A}\rangle=\langle\mathbf{a},\mathbf{a}\rangle+2\varepsilon\langle\mathbf{a},\mathbf{a}^*\rangle=1. \tag{2.18}$$

It follows that relations (2.11) and (2.18) are corresponding. Via this we have the following map (Study's Map) [25]: the set of all oriented lines in Euclidean space E^3 is in one-to-one correspondence with set of points of a dual unit sphere in D^3-space [25].

The dual representation of a line is simply the Plucker vector written as a dual unit vector. This vector is a point on a unit sphere in D^3 which is also the image of the Plucker quadratic in D^3. This representation has all the geometric structure offered by the Plucker coordinates with a simplified computational structure. This is a result of the Study's map, the points on the dual unit sphere representing lines in E^3. However, this representation is not unique. Each oriented line is represented by two points on the dual unit sphere. The antipodal points on the dual unit sphere represent the same line in E^3. Lines are points on the dual unit sphere and curves on it are ruled surfaces. The structure of the dual unit sphere offers some interesting insight into the geometry of the underlying ruled surface. This can be seen from a study of the properties of the curves on the dual unit sphere. Therefore, the terms dual curve (dual unit vector depending on a real parameter) and ruled surface are synonymous in this work.

This dualized form of line representation along with Study's map leads to a new interpretation of the scalar and vectorial products of two lines. For two directed lines $\mathbf{X}$ and $\mathbf{Y}$ the dual angle $\Theta=\theta+\varepsilon\theta^*$ combines the angle θ and the minimal distance θ^*. This gives rise to geometric interpretations of the following products of the dual unit vectors:

$$\langle\mathbf{X},\mathbf{Y}\rangle=\cos\Theta=\cos\theta-\varepsilon\theta^*\sin\theta. \tag{2.19}$$

The following special cases can be given:

1. If $\langle \mathbf{X}, \mathbf{Y} \rangle = 0$, then $\theta = \frac{\pi}{2}$ and $\theta^* = 0$; this means that the two lines **X** and **Y** meet at a right angle,
2. If $\langle \mathbf{X}, \mathbf{Y} \rangle =$ pure dual, then $\theta = \frac{\pi}{2}$ and $\theta^* \neq 0$; the lines **X** and **Y** are orthogonal skew lines,
3. If $\langle \mathbf{X}, \mathbf{Y} \rangle =$ pure real, then $\theta \neq \frac{\pi}{2}$ and $\theta^* = 0$; the lines **X** and **Y** intersect,
4. If $\langle \mathbf{X}, \mathbf{Y} \rangle = 1$, then $\theta = 0$ and $\theta^* = 0$; the lines **X** and **Y** are coincident (their senses are the same or opposite).

The vectorial product of **X** and **Y** is defined by

$$\mathbf{X} \times \mathbf{Y} = \mathbf{N} \sin \Theta,$$

where **N** represents a direct common perpendicular of the lines **X** and **Y**, and the signs of θ and $\theta^*$are related to the orientation of **N**. If oriented lines **X** and **Y** meet at right angle, then

$$\mathbf{Z} = \cos \Phi \mathbf{X} + \sin \Phi \mathbf{Y}$$

defines a line which is the image of **X** under a helical motion about the axis $\mathbf{X} \times \mathbf{Y}$ with dual angle Φ.

A ruled surface in the 3-dimensional Euclidean space E^3 is a one-parameter set of lines. The ruled surface is described as the line through a curve $\boldsymbol{\alpha} = \boldsymbol{\alpha}(u)$ and in the direction of $\mathbf{a} = \mathbf{a}(u)$ parametrized by

$$R : \mathbf{y}(u, t) = \boldsymbol{\alpha}(u) + t\mathbf{a}(u), \ t \in \mathbb{R}, \tag{2.20}$$

where $\boldsymbol{\alpha} = \boldsymbol{\alpha}(u)$ is its base curve and $\mathbf{a} = \mathbf{a}(u)$ is the unit vector along the direction of the generating lines of the surface, i.e. $\|\mathbf{a}\| = 1$. According to Study's map, Eq. (2.20) can be rewritten as:

$$\mathbf{A}(u) = \mathbf{a}(u) + \varepsilon \mathbf{a}^*(u) = \mathbf{a}(u) + \varepsilon \boldsymbol{\alpha}(u) \times \mathbf{a}(u).$$

Since the spherical image $\mathbf{a}(u)$ is a unit vector, the dual vector $\mathbf{A}(u)$ also has unit magnitude as is seen from the computations:

$$\begin{aligned}\langle \mathbf{A}, \mathbf{A} \rangle &= \langle \mathbf{a} + \varepsilon \boldsymbol{\alpha} \times \mathbf{a}, \mathbf{a} + \boldsymbol{\varepsilon} \boldsymbol{\alpha} \times \mathbf{a} \rangle \\ &= \langle \mathbf{a}, \mathbf{a} \rangle + 2\varepsilon \langle \mathbf{a}, \boldsymbol{\alpha} \times \mathbf{a} \rangle + \varepsilon^2 \langle \alpha \times \mathbf{a}, \boldsymbol{\alpha} \times \mathbf{a} \rangle = \langle \mathbf{a}, \mathbf{a} \rangle = 1.\end{aligned} \tag{2.21}$$

As a direct consequence of this representation, we can derive the properties of the one-parameter spatial motion of a line. Because this representation allows the geometry of ruled surfaces to be represented by the geometry of the one-parameter motion of a point on the dual unit sphere. A differentiable curve $\mathbf{A}(u)$ on a dual unit sphere in D^3, depending on a real parameter u represents a differentiable family of straight lines in Euclidean 3-space E^3 which we call a ruled surface. The lines **A** are the generators of the surface.

3 A new approach to a curve associated with a dual curve

The approach to a curve associated with a curve is presented in Euclidean 3-dimensional space E^3 [26]. We now develop this approach in the dual 3-space D^3 to a dual curve associated with a dual curve (ruled surface associated with a ruled surface) for meeting the requirement of instantaneous kinematic geometry of spatial motion because the movement of any line in a moving body is associated with the generator of the axode.

3.1 The Blaschke frame

Let $\mathbf{A}_1 = \mathbf{A}(u)$ be a dual curve on dual unit sphere $\langle \mathbf{A}, \mathbf{A} \rangle = 1$ in the dual 3-space D^3; as usual the Blaschke frame relative to $\mathbf{A}_1$ will be defined as the frame of which this line and the central normal $\mathbf{A}_2$ to the ruled surface at the central point of $\mathbf{A}_1$ are two edges. The third edge $\mathbf{A}_3$ is orthogonal to $\mathbf{A}_1$ and $\mathbf{A}_2$. Blaschke has shown that [24]:

$$\frac{d}{du}\begin{pmatrix}\mathbf{A}_1\\ \mathbf{A}_2\\ \mathbf{A}_3\end{pmatrix} = \begin{pmatrix}0 & P & 0\\ -P & 0 & Q\\ 0 & -Q & 0\end{pmatrix}\begin{pmatrix}\mathbf{A}_1\\ \mathbf{A}_2\\ \mathbf{A}_3\end{pmatrix}, \tag{3.1}$$

where

$$P = p + \varepsilon p^* = \left\|\mathbf{A}_1^{'}\right\|, \quad Q = q + \varepsilon q^* = \frac{\det(\mathbf{A}_1, \mathbf{A}_1^{'}, \mathbf{A}_1^{''})}{\left\|\mathbf{A}_1^{'}\right\|^2} \tag{3.2}$$

are called the Blaschke invariants of the dual curve $\mathbf{A}(u)$. Since P contains only first derivatives of the dual curve $\mathbf{A}(u)$, it is a first-order property of the curve; in particular, it is its dual speed. Similarly, Q is a second-order property of the dual curve $\mathbf{A}(u)$. Here the derivative with respect to u is denoted by a dash over a function symbol.

Consider the properties of two infinitesimally spaced generators, $\mathbf{A}(u)$ and $\mathbf{A}(u+\triangle u)$. The two generators are separated by a dual angle given by

$$dS = ds + \varepsilon ds^* = \left\|\mathbf{A}_1^{'}\right\| dt = P du, \tag{3.3}$$

where ds is the angle between the generators and ds^* is the minimal distance. We can define the distribution parameter as:

$$\lambda_a = \frac{ds^*}{ds} = \lim \frac{\triangle s^*}{\triangle s} = \frac{p^*}{p}. \tag{3.4}$$

From Eq. 3.4, it is obvious that, if the generators are parallel, the distribution parameter is infinite. In this case, the surface is a cylinder and the term dS is a pure dual number.

3.2 The Frenet frame

By means of Eq. 3.3, we have

$$\left\|\frac{d\mathbf{A}_1}{dS}\right\| = 1. \tag{3.5}$$

Hence we may say that $\mathbf{A} = \mathbf{A}_1(S)$ is a dual unit-speed curve. Let us denote $\mathbf{T}(S) = d\mathbf{A}_1/dS$ and call $\mathbf{T}(S)$ a unit tangent vector of $\mathbf{A}(S)$ at S. We define the dual curvature of $\mathbf{A} = \mathbf{A}_1(S)$ by $K = \varkappa + \varepsilon\varkappa^* = \left\|\frac{d^2A_1}{dS^2}\right\|$. If $K \neq 0$, the unit principal normal $\mathbf{N}(S)$ of the curve $\mathbf{A} = \mathbf{A}_1(S)$ at S is given by $d^2\mathbf{A}_1/dS^2 = K\mathbf{N}$. The unit vector $\mathbf{B} = \mathbf{T} \times \mathbf{N}$ is called a unit binormal vector of the curve $\mathbf{A} = \mathbf{A}_1(S)$ at S. The following Frenet formula now holds:

$$\frac{d}{dS}\begin{pmatrix}\mathbf{T}\\ \mathbf{N}\\ \mathbf{B}\end{pmatrix} = \begin{pmatrix}0 & K & 0\\ -K & 0 & T\\ 0 & -T & 0\end{pmatrix}\begin{pmatrix}\mathbf{T}\\ \mathbf{N}\\ \mathbf{B}\end{pmatrix}, \tag{3.6}$$

where $T(S) = \tau(S) + \varepsilon\tau^*(S)$ is the dual torsion of the curve $\mathbf{A} = \mathbf{A}_1(S)$ at S.

The Blaschke and Frenet frames have one common axis $\mathbf{T}$, so that a single dual angle $\Psi = \psi + \varepsilon\psi^*$ specifies completely their relative position. By definition, Ψ is measured in the negative sense of $\mathbf{T}$, so that:

$$\begin{pmatrix}\mathbf{A}_1\\ \mathbf{A}_2\\ \mathbf{A}_3\end{pmatrix} = \begin{pmatrix}0 & -\sin\Psi & \cos\Psi\\ 1 & 0 & 0\\ 0 & \cos\Psi & \sin\Psi\end{pmatrix}\begin{pmatrix}\mathbf{T}\\ \mathbf{N}\\ \mathbf{B}\end{pmatrix}. \tag{3.7}$$

3.3 A curve associated with a dual curve

During the motion of the Blaschke frame along $\mathbf{A}(u)$ there is a dual unit vector $\mathbf{X}$ that generates a dual curve $\mathbf{X}\,(u)$ different from $\mathbf{A}(u)$. Meanwhile, let the vector equation of $\mathbf{X}$ be given by

$$\mathbf{X} = X_1\mathbf{A}_1 + X_2\mathbf{A}_2 + X_3\mathbf{A}_3, \tag{3.8}$$

where $X_i = x_i + \varepsilon x_i^*(i = 1, 2, 3)$ are the dual coordinates of the dual point $\mathbf{X}$. Based on Eq. 3.1, the first derivative of $\mathbf{X}$ is given by

$$\mathbf{X}' = (X_1' - PX_2)\mathbf{A}_1 + (X_2' + PX_1 - QX_3)\mathbf{A}_2 + (X_3' + QX_2)\mathbf{A}_3. \tag{3.9}$$

By differentiating the above equation with respect to the motion parameter u, any further order of derivative of $\mathbf{X}$ will be obtained and the invariants of the dual curve $\mathbf{X}$ can be represented by the invariants of the dual curve $\mathbf{A}_1\,(u)$, which will be discussed later.

In particular, if the dual point $\mathbf{X}$ is fixed with respect to a fixed dual unit sphere in the dual 3-space D^3, we have $\mathbf{X}' = 0$, which implies

$$X_1' - PX_2 = 0, \quad X_2' - PX_1 + QX_3 = 0, \quad X_3' + QX_2 = 0. \tag{3.10}$$

In that case, the dual point $\mathbf{X}$ is called a fixed point and Eq. 3.10 is defined as a fixed-point condition of a dual curve associated with a dual curve.

4 The basic equations of one-parameter dual spherical motion

Consider two dual unit spheres K_m and K_f. Let $\mathbf{O}$ be the common center and two orthonormal dual coordinates frames $\{\mathbf{O}; \mathbf{L}_1, \mathbf{L}_2, \mathbf{L}_3\}$ and $\{\mathbf{O}; \mathbf{F}_1, \mathbf{F}_2, \mathbf{F}_3\}$ be rigidly linked to the dual unit spheres K_m and K_f, respectively. We suppose that $\{\mathbf{O}; \mathbf{F}_1, \mathbf{F}_2, \mathbf{F}_3\}$ is fixed, whereas the elements of the set $\{\mathbf{O}; \mathbf{L}_1, \mathbf{L}_2, \mathbf{L}_3\}$ are functions of a real parameter t (the time). Then we say that the dual unit sphere K_m moves with respect to the fixed dual unit sphere K_f. We may interpret this as follows: the dual unit sphere K_m is rigidly connected with $\{\mathbf{O}; \mathbf{L}_1, \mathbf{L}_2,\ \mathbf{L}_3\}$ and moves over the dual unit sphere K_f which is rigidly connected with $\{\mathbf{O}; \mathbf{F}_1, \mathbf{F}_2, \mathbf{F}_3\}$. This motion is called a one-parameter dual spherical motion and will be denoted by K_m/K_f. When the center of the dual unit sphere must remain fixed, the transformation groups in D^3, the image of the Euclidean motions, does not contain any translations. If the dual unit spheres K_m and K_f correspond to the line space H_m and H_f, respectively, then K_m/K_f corresponds to the one-parameter spatial motion H_m/H_f. Then H_m is the moving space with respect to the fixed space H_f.

Theorem 4.1 *The Euclidean motions in E^3 are represented in D^3 (the dual space) by dual orthogonal 3×3 matrices $A = (A_{ij})$ where $AA^t = I$, A_{ij} are dual numbers, and I is the 3×3 unit matrix.*

According to Theorem (4.1) the 3×3 dual matrix $A(t)$ of the motion K_m/K_f represents the one-parameter spatial motion H_m/H_f with the same parameter $t \in \mathbb{R}$.

The Lie algebra $L(O_{D^3})$ of the group GL of 3×3 positive orthogonal dual matrices A is the algebra of skew-symmetric 3×3 dual matrices

$$\Omega(t) = A'A^t = \begin{pmatrix} 0 & \Omega_3 & -\Omega_2 \\ -\Omega_3 & 0 & \Omega_1 \\ \Omega_2 & -\Omega_1 & 0 \end{pmatrix}, \tag{4.1}$$

where A' indicates the differentiation of A with respect to the real parameter t.

During the motion K_m/K_f the differential velocity vector of a fixed dual point X on K_m, analogous to the real spherical motion [4], is:

$$\frac{\mathrm{d}\mathbf{X}}{\mathrm{d}t} = \boldsymbol{\Omega} \times \mathbf{X}, \tag{4.2}$$

where $\mathbf{\Omega} = \boldsymbol{\omega} + \varepsilon\boldsymbol{\omega}^*$ is called the instantaneous Pfaffian vector of the motion K_m/K_f. The Pfaffian dual vector $\mathbf{\Omega}$ at the instant t of the one-parameter dual spherical motion K_m/K_f is analogues to the Darboux vector in the differential geometry of the space curves. In this case $\boldsymbol{\omega}$ and $\boldsymbol{\omega}^*$ correspond to the instantaneous rotational differential velocity vector and the instantaneous translational differential velocity vector of the corresponding spatial motion H_m/H_f, respectively. The dual number $\Omega = \omega + \varepsilon\omega^* = \|\mathbf{\Omega}\|$ is called the dual angular speed of the dual spherical motion K_m/K_f.

Let us define the following identification

$$\Omega = \begin{pmatrix} 0 & \Omega_3 & -\Omega_2 \\ -\Omega_3 & 0 & \Omega_1 \\ \Omega_2 & -\Omega_1 & 0 \end{pmatrix} \Leftrightarrow \begin{pmatrix} \Omega_1 \\ \Omega_2 \\ \Omega_3 \end{pmatrix} = \mathbf{\Omega}. \tag{4.3}$$

Consequently, we may form the vectors from $L(O_{D^3})$ in two ways as skew-symmeteric matrices or as vectors of the usual three-dimensional dual vector space. In what follows we will use both of these possibilities according to which of the two will be more advantageous in the given case. In analogy with the real spherical motion, we can introduce the following theorem:

Theorem 4.2 *For a one-parameter motion K_m/K_f, analogous to the real spherical motion, the following holds ([7]):*

(i) *The skew-symmetric 3×3 dual matrix (the dual vector function) determined by $\Omega_m(t) = A'A^tA$ is called the moving polode.*

(ii) *The skew-symmetric 3×3 dual matrix (the dual vector function) determined by*

$$\Omega_f = A'A^t, \tag{4.4}$$

is called the fixed polode.

(iii) *The moving and fixed polodes are related by*

$$\Omega_f(t) = adA(t)\Omega_m(t), \quad \text{where } adA\Omega_m = A\Omega_m A^t, \tag{4.5}$$

(iv) $\|\mathbf{\Omega}_f\| = \|\mathbf{\Omega}_m\|$,

(v) *The dual unit vectors*

$$\mathbb{R}_f(t) = \frac{\mathbf{\Omega}(t)}{\|\mathbf{\Omega}(t)\|} \quad \text{and} \quad \mathbb{R}_m(t) = \frac{\mathbf{\Omega}_m(t)}{\|\mathbf{\Omega}_m(t)\|}$$

are called the fixed axode and moving axodes of the motion H_m/H_f, respectively.

(vi) $\dfrac{d\mathbf{R}_f}{dt} = adA\dfrac{d\mathbf{R}_m}{dt} \Leftrightarrow \dfrac{d\mathbf{R}_f}{dt} = A\dfrac{d\mathbf{R}_m}{dt}A^t.$

A detailed exposition of this material can be found, for instance, in [7]. The following Lemma will be useful:

Lemma 4.1 *Let* **A** *and* **B** *be arbitrary dual unit vectors, and* $\mathbf{C} = \mathbf{A} \times \mathbf{B}$. *The vectorial product-matrix of* **A** *and* **B** *is given by*

$$C = BA^t - AB^t. \tag{4.6}$$

Proof This follows by noticing that, for any dual unit vector $\mathbf{X}$,

$$\mathbf{C} \times \mathbf{X} = (\mathbf{A} \times \mathbf{B}) \times \mathbf{X} = \mathbf{B}(A^tX) - \mathbf{A}(B^tX), \tag{4.7}$$

which leads to

$$CX = (BA^t - AB^t)X, \tag{4.8}$$

in view of the identification in (4.3). Therefore, the Lemma is proved.

Let us use the superscript (m) or subscript m indicating that a dual unit vector or a dual invariant belongs to $\mathbf{R}_m$, and also the superscript (f) or subscript f indicating that for $\mathbf{R}_f$. Applying this to Study's map we have that the dual unit vector $\mathbf{R}_f$ is a function of t; it represents the locus of the instantaneous screw axis (*ISA* for short) *ISA* on K_f. This curve corresponds to a ruled surface in H_f and is called fixed axode. This fixed axode is made up those lines in the fixed space H_f which at some instant coincide with a line in the moving space having zero dual velocity. Likewise, the dual unit vector $\mathbf{R}_m$ is a function of t, which represents the locus of the *ISA* on K_m. This locus corresponds to a ruled surface in H_m and is called the moving axode.

During the motion K_m/K_f, the differentiable curve

$$t \in \mathbb{R} \to \mathbf{R}_m(t) \in K_m, \tag{4.9}$$

represents a differentiable family of straight lines or the moving axode. We now define an orthonormal moving frame along this dual curve as follows:

$$\mathbf{R}_1^m = \mathbf{R}_m(t), \quad \mathbf{R}_2^m = \left(\frac{\mathrm{d}\mathbf{R}_m}{\mathrm{d}t}\right)\left\|\frac{\mathrm{d}\mathbf{R}_m}{\mathrm{d}t}\right\|^{-1}, \quad \mathbf{R}_3^m = \mathbf{R}_1^m \times \mathbf{R}_2^m. \tag{4.10}$$

This frame is called the Blaschke frame, and the corresponding lines intersect at the striction point of the axode $\mathbf{R}_1^m = \mathbf{R}_m(t)$. $\mathbf{R}_3^m$ and $\mathbf{R}_2^m$ are known as the central tangent and the central normal of the ruled surface $\mathbf{R}_m = \mathbf{R}_1^m(t)$, respectively. Let K_r^m be a dual unit sphere generated by the set $\{\mathbf{O};\ \mathbf{R}_1^m,\ \mathbf{R}_2^m,\ \mathbf{R}_3^m\}$. Therefore, the motion K_r^m/K_m is described by

$$\frac{\mathrm{d}\mathbf{R}_m}{\mathrm{d}t} = B_m\mathbf{R}_m, \tag{4.11}$$

where

$$B_m = \begin{pmatrix} 0 & P_m & 0 \\ -P_m & 0 & Q_m \\ 0 & -Q_m & 0 \end{pmatrix} \quad \text{and} \quad \mathbf{R}_m = \begin{pmatrix} \mathbf{R}_1^m \\ \mathbf{R}_2^m \\ \mathbf{R}_3^m \end{pmatrix}, \tag{4.12}$$

and the dual functions

$$P_m = p_m + \varepsilon p_m^* = \left\|\frac{\mathrm{d}\mathbf{R}_m}{\mathrm{d}t}\right\|, \quad Q_m = q_m + \varepsilon q_m^* = \frac{\det(\mathbf{R}_1^m, \frac{\mathrm{d}\mathbf{R}_1^m}{\mathrm{d}t}, \frac{\mathrm{d}^2\mathbf{R}_1^m}{\mathrm{d}t^2})}{P_m^2}, \tag{4.13}$$

are called the Blaschke invariants of the moving axode. The integrals $\int P_m \mathrm{d}t$, and $\int Q_m \mathrm{d}t$ are the dual arc-length of the dual curves $\mathbf{R}_1^m(t)$ and $\mathbf{R}_3^m(t)$, respectively. The striction (central) point, $\mathbf{C}^m(t)$, satisfies:

$$\mathbf{C}^m \times \mathbf{r}_i^m = \mathbf{r}_i^{*m}, \ (i = 1, 2, 3). \tag{4.14}$$

Differentiating the three Eqs. in 4.14 and using (4.12), we have:

$$\frac{\mathrm{d}\mathbf{C}_{(m)}}{\mathrm{d}t} = q_m^*\mathbf{r}_1^m + p_m^*\mathbf{r}_3^m. \tag{4.15}$$

If we use the arc length of the striction curve as the motion parameter, then the dual functions in (4.13) obey

$$P_m = p_m + \varepsilon \sin \sigma_m, \quad Q_m = q_m + \varepsilon \cos \sigma_m\ , \tag{4.16}$$

where σ_m is the striction angle measuring the deviation of the generating lines of $\mathbf{R}^m(t)$ from the striction curve. The distribution parameter of the moving axode is

$$\lambda_m = \frac{p_m^*}{p_m} = \frac{\sin \sigma_m}{p_m}. \tag{4.17}$$

Now, we shall carry on as above. During the one-parameter dual spherical motion K_m/K_f, the *ISA* on K_f generates the fixed polode which admits the Blaschke frame

$$\mathbf{R}_1^f = \mathbf{R}_f(t), \quad \mathbf{R}_2^f = \left(\frac{\mathrm{d}\mathbf{R}_f(t)}{\mathrm{d}t}\right)\left\|\frac{\mathrm{d}\mathbf{R}_f}{\mathrm{d}t}\right\|^{-1}, \quad \mathbf{R}_3^f = \mathbf{R}_1^f \times \mathbf{R}_2^f. \tag{4.18}$$

Likewise the set $\{\mathbf{O};\ \mathbf{R}_1^{(f)},\ \mathbf{R}_2^{(f)}, \mathbf{R}_3^{(f)}\}$ defines a dual unit sphere K_r^f, and the motion K_r^f/K_f is given by

$$\frac{\mathrm{d}\mathbf{R}_f}{\mathrm{d}t} = B_f\mathbf{R}_f, \tag{4.19}$$

where

$$B_f = \begin{pmatrix} 0 & P_f & 0 \\ -P_f & 0 & Q_f \\ 0 & -Q_f & 0 \end{pmatrix} \quad \text{and} \quad \mathbf{R}_f = \begin{pmatrix} \mathbf{R}_1^f \\ \mathbf{R}_2^f \\ \mathbf{R}_3^f \end{pmatrix}. \tag{4.20}$$

Similarly, the dual functions

$$P_f = p_f + \varepsilon p_f^* = \left\| \frac{\mathrm{d}\mathbf{R}_f}{\mathrm{d}t} \right\|, \quad Q_f = q_f + \varepsilon q_f^* = \frac{\det(\mathbf{R}_1^f, \frac{\mathrm{d}\mathbf{R}_1^f}{\mathrm{d}t}, \frac{\mathrm{d}^2\mathbf{R}_1^f}{\mathrm{d}t^2})}{P_f^2}, \tag{4.21}$$

are the Blaschke invariants of the fixed axode. The integrals $\int P_f \mathrm{d}t$, and $\int Q_f \mathrm{d}t$ are the dual arc-length of the dual curves $\mathbf{R}_1^f(t)$ and $\mathbf{R}_3^f(t)$, respectively. And the striction curve is given by

$$\frac{\mathrm{d}\mathbf{C}^f}{\mathrm{d}t} = q_f^*\mathbf{r}_1^f + p_f^*\mathbf{r}_3^f, \tag{4.22}$$

Analogously, the dual functions in (4.21) are:

$$P_f = p_m + \varepsilon \sin\sigma_f,\ Q_f = q_f + \varepsilon \cos\sigma_f\ , \tag{4.23}$$

where σ_f is the striction angle of the lines of $\mathbf{R}_f(t)$ with the striction curve. Hence, the distribution parameter of the fixed axode is

$$\lambda_f = \frac{p_f^*}{p_f} = \frac{\sin\sigma_f}{p_f}. \tag{4.24}$$

Theorem 4.3 *Under the above notation, for the Blaschke invariants of the axodes, we have*

$$P_m = P_f, \quad Q_f - Q_m = \|\boldsymbol{\Omega}\|\,. \tag{4.25}$$

Proof This follows by noticing that, from (4.12), (4.20) and (vi) in Theorem (4.2), we have

$$\frac{\mathrm{d}\mathbf{R}_m}{\mathrm{d}t} = P_m\mathbf{R}_2^m, \quad \frac{\mathrm{d}\mathbf{R}_f}{\mathrm{d}t} = P_f\mathbf{R}_2^f. \tag{4.26}$$

This implies

$$P_f^2 = \left\langle \frac{\mathrm{d}\mathbf{R}_f}{\mathrm{d}t}, \frac{\mathrm{d}\mathbf{R}_f}{\mathrm{d}t} \right\rangle = \left\langle adA(t)\frac{\mathrm{d}\mathbf{R}_m}{\mathrm{d}t}, adA(t)\frac{\mathrm{d}\mathbf{R}_m}{\mathrm{d}t} \right\rangle = \left\langle \frac{\mathrm{d}\mathbf{R}_m}{\mathrm{d}t}, \frac{\mathrm{d}\mathbf{R}_m}{\mathrm{d}t} \right\rangle = P_m^2, \tag{4.27}$$

since the scalar product is invariant under adA. Since $P_m > 0, P_f > 0$, we have $P_m{=}P_f$. From the equation

$$\frac{\mathrm{d}\mathbf{R}_f}{\mathrm{d}t} = adA\left(\frac{\mathrm{d}\mathbf{R}_m}{\mathrm{d}t}\right), \tag{4.28}$$

we have

$$P_f\mathbf{R}_2^f = adA(P_m\mathbf{R}_2^m) \Rightarrow \mathbf{R}_2^f = adA(\mathbf{R}_2^m), \tag{4.29}$$

since adA is linear. Differentiation of the last equation gives

$$\begin{aligned}\frac{d\mathbf{R}_2^f}{dt} &= \frac{d}{dt}(A\mathbf{R}_2^m A^t) = \frac{dA}{dt}\mathbf{R}_2^m A^t + A\mathbf{R}_2^m\frac{dA^t}{dt} + A\frac{d\mathbf{R}_2^m}{dt}A^t\\ &= \frac{dA}{dt}A^tA\mathbf{R}_2^m A^t + A\mathbf{R}_2^m A^tA\frac{dA^t}{dt} + adA\left(\frac{d\mathbf{R}_2^m}{dt}\right)\\ &= \Omega_f\mathbf{R}_2^f - \mathbf{R}_2^f\Omega_f + adA\left(\frac{d\mathbf{R}_2^m}{dt}\right) = \Omega_f\mathbf{R}_3^f + adA\left(\frac{d\mathbf{R}_2^m}{dt}\right),\end{aligned} \tag{4.30}$$

where we have used the Eqs. 4.5 and 4.6. Thus, substituting now from (4.11), and (4.19) in the last relation, we have

$$-P_f\mathbf{R}_1^f + Q_f\mathbf{R}_3^f = \Omega\mathbf{R}_3^f + adA(-P_m\mathbf{R}_1^m + Q_m\mathbf{R}_3^m) \tag{4.31}$$

from which we get

$$-P_f\mathbf{R}_1^f + Q_f\mathbf{R}_3^f = \Omega\mathbf{R}_3^f - P_m adA(\mathbf{R}_1^m) + Q_m adA(\mathbf{R}_3^m). \tag{4.32}$$

which leads to

$$Q_f\mathbf{R}_3^f = \Omega_f\mathbf{R}_f^3 + Q_m\mathbf{R}_3^f. \tag{4.33}$$

We have used

$$adA\mathbf{R}_3^m = \mathbf{R}_3^f, \quad adA\mathbf{R}_1^m = \mathbf{R}_1^f. \tag{4.34}$$

Hence

$$Q_f\mathbf{R}_3^f = \Omega_f\mathbf{R}_3^f + Q_m\mathbf{R}_3^f. \tag{4.35}$$

Therefore, we have

$$Q_f - Q_m = \Omega. \tag{4.36}$$

This completes the proof of the theorem. As a result, the following corollary can be given

Corollary 4.1 *At any instant t, during the one-parameter spherical motion K_m/K_f, the moving polode is in contact with the fixed polode along the ISA in the first order at any instant t. The common distribution parameter of the axodes is*

$$\lambda := \lambda_m = \lambda_f = \frac{p^*}{p}. \tag{4.37}$$

Let K_r be a dual unit sphere generated by the right-handed dual system $\{\mathbf{O};\ \mathbf{R} = \mathbf{R}_1,\ \mathbf{R}_2,\ \mathbf{R}_3\}$ which is defined as follows: $\mathbf{R}_1(t) = \mathbf{r}_1(t) + \varepsilon\mathbf{r}_1^*(t)$ as the instantaneous screw axis ISA, $\mathbf{R}_3(t) = \mathbf{r}_3(t) + \varepsilon\mathbf{r}_3^*(t)$ is the common perpendicular of $\mathbf{R}_1(t)$ and $\mathbf{R}_1(t + dt)$, and $\mathbf{R}_2(t) = \mathbf{r}_2(t) + \varepsilon\mathbf{r}_2^*(t) = \mathbf{R}_1 \times \mathbf{R}_2$. This frame is called relative frame and it is fully determined by the first-order properties of the dual spherical motion K_m/K_f. The dual unit vectors $\mathbf{R}_1$, $\mathbf{R}_2$, and $\mathbf{R}_3$ correspond to three concurrent mutually orthogonal lines in the Euclidean 3-space E^3. Their point of intersection is the striction point of the moving and fixed axodes. The lines $\mathbf{R}_2$ and $\mathbf{R}_3$ are called the central normal and the central tangent of the axodes at the striction point, respectively. Then, the derivative equations of the dual spherical motions K_r/K_m and K_r/K_f, respectively, are:

$$\frac{d\mathbf{R}}{dt}\Big| m = C(M)\mathbf{R}, \quad \mathbf{R} = \begin{pmatrix}\mathbf{R}_1\\ \mathbf{R}_2\\ \mathbf{R}_3\end{pmatrix}, \quad C(M) = \begin{pmatrix}0 & P & 0\\ -P & 0 & Q_m\\ 0 & -Q_m & 0\end{pmatrix}, \tag{4.38}$$

and

$$\left.\frac{\mathrm{d}\mathbf{R}}{\mathrm{d}t}\right|_f = C(F)\mathbf{R}, \quad C(F) = \begin{pmatrix} 0 & P & 0 \\ -P & 0 & Q_f \\ 0 & -Q_f & 0 \end{pmatrix}, \tag{4.39}$$

where

$$P = p + \varepsilon p^* \quad \text{and} \quad Q_m = q_m + \varepsilon q_m^*, \;\; Q_f = q_f + \varepsilon q_f^*, \tag{4.40}$$

are the invariants of the motion K_m/K_f.

5 The approach to a ruled surface associated with the axodes

In the above section, the dual invariants of the polodes are defined. Now, we will analyse their geometrical and kinematic meanings. For this purpose, consider a dual point $\mathbf{X}$ such that its coordinates are:

$$\sum_{i=1}^{3}\mathbf{X}_i^2 = 1, \;\mathbf{X} = X^t\mathbf{R}, \quad \mathbf{X} = \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}. \tag{5.1}$$

If $\mathbf{X}$ is a function of t, the velocity of $\mathbf{X}$ at the instant t with respect to the moving space H_m is

$$\left.\frac{\mathrm{d}\mathbf{X}}{\mathrm{d}t}\right|_m = \frac{\mathrm{d}X^t}{\mathrm{d}t}\mathbf{R} + X^t\left.\frac{\mathrm{d}\mathbf{R}}{\mathrm{d}t}\right|_m. \tag{5.2}$$

Similarly, we obtain the velocity of $\mathbf{X}$ with respect to the fixed space H_f as follows:

$$\left.\frac{\mathrm{d}\mathbf{X}}{\mathrm{d}t}\right|_f = \frac{\mathrm{d}X^t}{\mathrm{d}t}\mathbf{R} + X^t\left.\frac{\mathrm{d}\mathbf{R}}{\mathrm{d}t}\right|_f, \tag{5.3}$$

or from (4.38) and (4.39), we get

$$\left.\frac{\mathrm{d}\mathbf{X}}{\mathrm{d}t}\right|_m = \left(\frac{\mathrm{d}X^t}{\mathrm{d}t} + X^t C(M)\right)\mathbf{R}, \tag{5.4}$$

and

$$\left.\frac{\mathrm{d}\mathbf{X}}{\mathrm{d}t}\right|_f = \left(\frac{\mathrm{d}X^t}{\mathrm{d}t} + X^t C(F)\right)\mathbf{R}. \tag{5.5}$$

In particular, if the line $\mathbf{X}$ is fixed relative to the moving space H_m, then the derivative $\frac{\mathrm{d}\mathbf{X}}{\mathrm{d}t}\big|_m = 0$. This complies with the fixed dual-point condition of a dual curve associated with a dual curve, that is:

$$\frac{\mathrm{d}X^t}{\mathrm{d}t} = X^t C(M)^t, \tag{5.6}$$

where

$$C(M)^t + C(M) = 0. \tag{5.7}$$

Now, suppose that $\mathbf{X}$ is fixed relative to the moving space H_m and let us calculate its velocity with respect to the fixed space H_f. Then we substitute (5.7) in (5.6) and obtain

$$\frac{\mathrm{d}\mathbf{X}}{\mathrm{d}t} = X^t(C(F) - C(M))\mathbf{R}. \tag{5.8}$$

Let us define a new matrix $C(R)$ by

$$C(R) = C(F) - C(M). \tag{5.9}$$

Then (5.8) can be rewritten as:

$$\frac{d\mathbf{X}}{dt} = X^t C(R)\mathbf{R}. \tag{5.10}$$

In view of (5.9), the matrix $C(R)$ is skew-symmetric, i.e.,

$$C(R)^t + C(R) = 0, \tag{5.11}$$

and therefore possesses an axial dual vector $\mathbf{D} = \mathbf{d} + \varepsilon\mathbf{d}^*$ such that

$$C(R)\mathbf{X} = \mathbf{D} \times \mathbf{X}. \tag{5.12}$$

Thus Eq. 5.10 can be written as follow:

$$\frac{d\mathbf{X}}{dt} = \mathbf{D} \times \mathbf{X}, \quad \mathbf{D} = \mathbf{D}^{(f)} - \mathbf{D}^{(m)} = \Omega\mathbf{R}_1, \tag{5.13}$$

where

$$\|\mathbf{\Omega}\| = \Omega = \omega + \varepsilon\omega^*. \tag{5.14}$$

From Eq. 5.13, it follows that the acceleration of $\mathbf{X}$ is given by

$$\frac{d^2\mathbf{X}}{dt^2} = \Omega'\mathbf{R}_1 \times \mathbf{X} + \Omega\mathbf{R}_1' \times \mathbf{X} + \Omega^2\mathbf{R}_1 \times (\mathbf{R}_1 \times \mathbf{X}),$$

which implies that

$$\frac{d^2\mathbf{X}}{dt^2} = X_3 P\Omega\mathbf{R}_1 - (X_2\Omega^2 + X_3\Omega')\mathbf{R}_2 + (X_2\Omega' - X_1 P\Omega - \Omega^2 X_3)\mathbf{R}_3. \tag{5.15}$$

The real part ω and the dual part ω^* correspond, respectively, to rotational and translational motions when the moving axode rolls about and slides along a common ruling. Hence the following corollary can be given

Corollary 5.1 *At any instant t, during the one-parameter spatial motion H_m/H_f for a non-vanishing distribution parameter λ of the axodes, the pitch of the motion can be expressed as:*

$$h = \frac{\omega^*}{\omega} = \frac{q_f^* - q_m^*}{q_f - q_m} = \lambda\left(\frac{\cot\sigma_f - \cot\sigma_m}{\cot\theta_f - \cot\theta_m}\right), \tag{5.16}$$

where σ_f, σ_m are the striction angles and θ_f, θ_m are the apex angle of the director cone of revolution of the axodes.

5.1 The inflection line congruence

According to Study's map, four independent parameters define an oriented line, so it is possible to intersect any of two line complexes, and obtain a finite number of lines with associated properties. The intersection of three of the line complexes yields a ruled surface. Now it is useful to study the ruled surface generated by the associated line $\mathbf{X}$. For general purposes we define a dual frame moving along the dual curve $\mathbf{X}(t)$ on K_f. This curve corresponds to a ruled surface in the fixed space H_f. The Blaschke frame along $\mathbf{X}(t)$ is defined as follows:

$$\mathbf{E}_1 = \mathbf{X} = X_1\mathbf{R}_1 + X_2\mathbf{R}_2 + X_3\mathbf{R}_3, \tag{5.17}$$

$$\mathbf{E}_2 = \frac{\mathbf{X}'}{\|\mathbf{X}'\|} = \frac{-X_3\mathbf{R}_2 + X_2\mathbf{R}_3}{\sqrt{1 - X_1^2}}, \tag{5.18}$$

$$\mathbf{E}_3 = \frac{\mathbf{X}'}{\|\mathbf{X}'\|} \times \mathbf{E}_2 = \frac{(1 - X_1^2)\mathbf{R}_1 - X_1X_2\mathbf{R}_2 - X_1X_3\mathbf{R}_3}{\sqrt{1 - X_1^2}}. \tag{5.19}$$

The dual unit vectors $\mathbf{E}_1$, $\mathbf{E}_2$, and $\mathbf{E}_3$ correspond to three concurrent mutually orthogonal lines in E^3. Their point of intersection is the striction point of the ruling $\mathbf{X}$. By construction, the Blaschke formula is

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\mathbf{E}_1\\ \mathbf{E}_2\\ \mathbf{E}_3\end{pmatrix} = \begin{pmatrix}0 & P_x & 0\\ -P_x & 0 & Q_x\\ 0 & -Q_x & 0\end{pmatrix}\begin{pmatrix}\mathbf{E}_1\\ \mathbf{E}_2\\ \mathbf{E}_3\end{pmatrix}, \tag{5.20}$$

where

$$P_x = p_x + \varepsilon p_x^* = \Omega\sqrt{1 - X_1^2}, \quad Q_x = q_x + \varepsilon q_x^* = \Omega X_1 + \frac{PX_3}{1 - X_{\mathbf{1}}^2}, \tag{5.21}$$

are the Blaschke invariants of the dual curve $\mathbf{X}(t)$.

The distribution parameter of the ruled surface generated by the associated line $\mathbf{X}$ can be given by

$$\lambda_x = \frac{x_2x_2^* + x_3x_3^* + h(x_2^2 + x_3^2)}{(x_2^2 + x_3^2)}. \tag{5.22}$$

Equation 5.22 can be used to identify those associated lines of the moving axode that trace ruled surfaces having the same distribution parameter. This set of lines is called a *line complex* and is defined by the equation

$$x_2x_2^* + x_3x_3^* + (h - \lambda_x)(x_2^2 + x_3^2) = 0, \tag{5.23}$$

Equation 5.23 represents a quadratic line complex. As a result the following theorem can be given:

Theorem 5.1 *During the one-parameter spatial motion H_m/H_f, consider a set of lines associated with the moving axode and these lines are generators of ruled surfaces having the same distribution parameter in the fixed space H_f. Then this set of lines belong to a quadratic line complex.*

Now, let $\mathbf{p}(x, y, z)$ be the position vector of an arbitrary point on the associated line $\mathbf{X}$, then

$$\mathbf{x}^* = \mathbf{p} \times \mathbf{x},$$

or

$$x_1^* = x_3y - x_2z,\ x_2^* = x_1z - x_3x,\ x_3^* = x_2x - x_1y. \tag{5.24}$$

Then Eq. 5.23 take the form

$$-x_1x_3y + x_1x_2z + (h - \lambda_x)(x_2^2 + x_3^2) = 0. \tag{5.25}$$

This equation shows that the associated lines $\mathbf{X}$ of the moving axode that trace ruled surfaces with the same distribution parameter lie on a plane parallel to the *ISA* of the one-parameter spatial motion H_m/H_f.

From Eq. 5.25, we have two different cases: In the case of $\lambda_x = h$ the distribution parameter is associated with the lines in planes passing through the *ISA*. In the case of $\lambda_x = 0$, the associated line $\mathbf{X}$ of the moving axode, at instant t, generate a developable ruled surface, and Eq. 5.25 reduces to

$$-x_1x_3y + x_1x_2z + h(x_2^2 + x_3^2) = 0. \tag{5.26}$$

In this case the generating line $\mathbf{X}$ and its neighboring $\mathbf{X}(t+\mathrm{d}t)$ meet at the edge of regression of the ruled surface, i.e. the tangent lines of the edge of regression are those lines. Then we have the following theorem:

Theorem 5.2 *During the one-parameter spatial motion H_m/H_f, if associated lines of the moving axode generates developable ruled surfaces in H_f, these lines are included in a special quadratic line complex which are identical to line complex of the tangent lines of edge points in H_f.*

The invariants $P_x = p_x + \varepsilon p_x^*$, and $Q_x = q_x + \varepsilon q_x^*$ provide a kinematic interpretation of the Blaschke frame. To carry out this, we define the dual vector

$$\mathbf{D} = Q_x\mathbf{E}_1 + P_x\mathbf{E}_3, \tag{5.27}$$

known as Darboux's vector. According to this vector, at any instant t, the dual angular velocity vector of the Blaschke frame with respect to itself has a component Q_x about $\mathbf{E}_1$ and P_x about $\mathbf{E}_3$. $\|\mathbf{D}\| = \sqrt{Q_x^2 + P_x^2} = \omega_x + \varepsilon\omega_x^*$ is the angular speed of $\mathbf{E}_1$ about the Darboux vector;

$$\omega_x = \sqrt{q_x^2 + p_x^2}, \quad \omega_x^* = \frac{p_x p_x^* + q_x q_x^*}{\sqrt{q_x^2 + p_x^2}}, \tag{5.28}$$

are the rotational angular speed and translational angular speed of $\mathbf{E}_1$, respectively. The pitch of $\mathbf{E}_1$ along the Darboux vector is

$$h_x = \frac{\omega_x^*}{\omega_x} = \frac{p_x p_x^* + q_x q_x^*}{q_x^2 + p_x^2}. \tag{5.29}$$

Now we define the dual unit vector

$$\mathbf{U} = \frac{\mathbf{D}}{\|\mathbf{D}\|} = \frac{Q_x\mathbf{E}_1 + P_x\mathbf{E}_3}{\sqrt{Q_x^2 + P_x^2}}, \tag{5.30}$$

where $\mathbf{U}$ is the Disteli axis of motion of the line $\mathbf{E}_1$ in the Blaschke frame. From Eq. 5.30, the Disteli axis is parallel to the tangent plane of the ruled surface $\mathbf{X} = \mathbf{X}(t)$, and is orthogonal to the central normal $\mathbf{E}_2$. Therefore the *ISA* of the one-parameter spherical motion K_m/K_f and the Disteli axis lie on a single great dual circle determined by the intersection of the $\mathbf{E}_1\mathbf{E}_3$-plane with the dual unit sphere K_f. Let $\Delta = \delta + \varepsilon\delta^*$ be the dual angle between the Disteli axis and the associated line $\mathbf{X}$; then we have

$$\mathbf{U} = \cos\Delta\mathbf{E}_1 + \sin\Delta\mathbf{E}_3. \tag{5.31}$$

Note that $\Delta = \delta + \varepsilon\delta^*$ is the *dual spherical radius of curvature.* It follows from the differentiation of (5.31) that:

$$\mathbf{U}' = (-\sin\Delta\mathbf{E}_1 + \cos\Delta\mathbf{E}_3)\Delta' + (P_x\cos\Delta - Q_x\sin\Delta)\mathbf{E}_2, \tag{5.32}$$

which leads to

$$\cot\Delta = \frac{Q_x}{P_x}. \tag{5.33}$$

This equation gives the relationship between the d*ual spherical curvature* Σ and the dual spherical radius of curvature as

$$\Sigma = \sigma + \varepsilon\sigma^* = \cot\Delta. \tag{5.34}$$

From the definition of the dual spherical radius of curvature we see that, if $\delta = \pi/2$ and $\delta^* = 0$, then $\Sigma = 0$. In this case $\mathbf{E}_1, \mathbf{E}_2$, and $\mathbf{U}$ are mutually orthogonal lines that intersect at the striction point of the ruled surface $\mathbf{X} = \mathbf{X}(t)$. When $\mathbf{U}$ and $\mathbf{E}_3$ line up, the ruled surface $\mathbf{X} = \mathbf{X}(t)$ is defined by a screw motion of the line $\mathbf{E}_1$ about the line $\mathbf{U}$ to the second order; usually this screw motion is only defined to first order; the distribution parameter is the pitch of this motion , i.e., $\lambda_x = h_x$. This is the analog for a ruled surface of an inflection point on a curve; for this reason the set of lines defined by $\Sigma = 0$ is called inflection line congruence. Hence, one can see that an explicit equation for inflection line congruence is obtained by the equation:

$$\Sigma = 0. \tag{5.35}$$

We substitute from Eq. 5.21 in (5.53) and obtain

$$\Omega X_1(X_2^2 + X_3^2) + PX_3 = 0, \tag{5.36}$$

which is a dual spherical curve of third degree. If we calculate the real and dual parts of (5.36), we have

$$\left.\begin{array}{l}\omega x_1(x_2^2 + x_3^2) + px_3 = 0,\\ (\omega^* x_1 + x_1^*\omega)(x_2^2 + x_3^2) + 2\omega x_1(x_2x_2^* + x_3x_3^*) + px_3^* + p^*x_3 = 0.\end{array}\right\} \tag{5.37}$$

The real part of Eq. 5.37 identifies the spherical cone of the motion H_m/H_f. Associated with the direction of a line on the spherical cone, there is a plane of lines defined by the dual part of Eq. 5.37. The inflection-line congruence consists of a set of a plane of lines, each of which is associated with a direction of the spherical cone of the motion H_m/H_f. Note that the Plucker coordinates $x_i,\ x_i^*(i = 1,\ 2,\ 3)$ satisfy the equations

$$x_1^2 + x_2^2 + x_3^2 = 1, \quad x_1x_1^* + x_2x_2^* + x_3x_3^* = 0. \tag{5.38}$$

Each equation of (5.37) with the Eqs. 5.38 represent a cubic line complex whose common lines form the inflection-line congruence. Hence, we summarize this result in the following theorem:

Theorem 5.3 *In the one-parameter spatial motion H_m/H_f, consider a set of associated lines of the moving axode, such that, at the instant t, each of these lines has an analog of an inflection point on a curve. Then this set of lines form a line congruence which is the intersection of two cubic line complexes.*

6 The line trajectories and Disteli formulae

The Eular–Savary equation in planar kinematics relates the position of a point to the position of its center of curvature and is the basis for a graphical construction yielding one given the other. Disteli [27] succeeded in presenting a set of formulas which generalizes the Eular–Savary equation to the spatial kinematics of line trajectories. This section gives a new method for deriving a new Disteli formula of spatial kinematics by means of a dual angle. This means that we seek an oriented line in the moving space H_m with a fixed dual angle with respect to a given line in the fixed space H_f. Hence, for the instantaneous fixed line $\mathbf{X}$ of the motion H_m/H_f, we introduce the dual angles $\Theta = \theta + \theta^*$, and $\Phi = \varphi + \varepsilon\varphi^*$ (see Fig. 1) to identify the direction of $\mathbf{X}$. Since $\mathbf{X}$ is a dual unit vector, we can write out the components of $\mathbf{X}$ in the following form:

$$\mathbf{X} = \cos\Theta\mathbf{R}_1 + \sin\Theta\mathbf{L}; \quad \mathbf{L} = \cos\Phi\mathbf{R}_2 + \sin\Phi\mathbf{R}_3\ . \tag{6.1}$$

This choice of coordinates is such that $\Phi = \varphi + \varepsilon\varphi^*$ is the dual angle between the central normal $\mathbf{E}_2$ of $\mathbf{X}(t)$ and $\mathbf{R}_2$ measured about the *ISA*; this means that a screw motion through an angle φ about the *ISA* and distance φ^* along it caries $\mathbf{R}_2$ into the central normal $\mathbf{E}_2$ of $\mathbf{X}(t)$. The dual angle $\Theta = \theta + \theta^*$ defines the position of $\mathbf{X}$ relative to the *ISA* of the motion H_m/H_f.

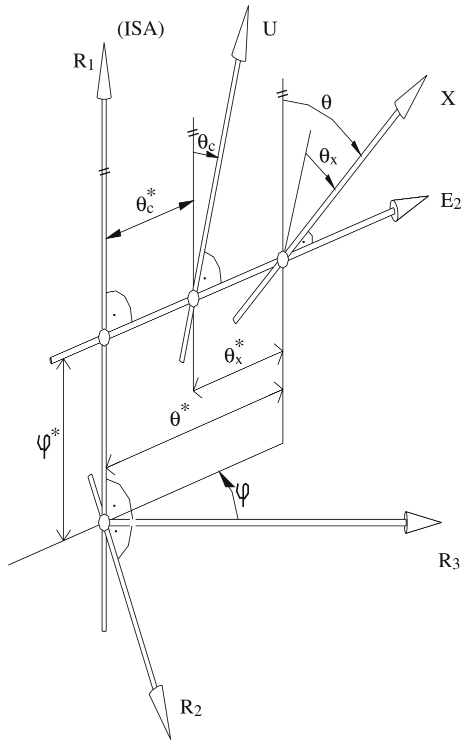
A similar set of coordinates may be used to identify the Disteli axis, $\mathbf{U}$, of the surface $\mathbf{X}(t)$ at $t = t_0$. Since the central normal $\mathbf{E}_2$ is also normal to the Disteli axis, it is identified by the same dual angle Φ about the *ISA* of the motion H_m/H_f. Denoting its dual angle with the *ISA* by $\Theta_c = \theta_c + \varepsilon\theta_c^*$, we have

$$\mathbf{U} = \cos\Theta_c\mathbf{R}_1 + \sin\Theta_c\cos\Phi\mathbf{R}_2 + \sin\Theta_c\sin\Phi\mathbf{R}_3. \tag{6.2}$$

From Eqs. 6.1 and 6.2 we require

$$\langle\mathbf{X}, \mathbf{U}\rangle = \cos(\Theta_c - \Theta) \tag{6.3}$$

Fig. 1 The moved line X and its Distelli axis U



and such that $\mathbf{U}$, and $(\Theta_c - \Theta)$ remain constant up to the second order at $t = t_0$, i.e.,

$$\left.\frac{\mathrm{d}(\Theta_c - \Theta)}{\mathrm{d}t}\right|_{t=t_0} = 0, \quad \left.\frac{\mathrm{d}\mathbf{U}}{\mathrm{d}t}\right|_{t=t_0} = \mathbf{0}, \tag{6.4}$$

and

$$\left.\frac{\mathrm{d}^2(\Theta_c - \Theta)}{\mathrm{d}t^2}\right|_{t=t_0} = 0, \quad \left.\frac{\mathrm{d}^2\mathbf{U}}{\mathrm{d}t^2}\right|_{t=t_0} = \mathbf{0}.$$

We have for the first order

$$\left\langle \frac{\mathrm{d}\mathbf{X}}{\mathrm{d}t}, \mathbf{U} \right\rangle = 0, \tag{6.5}$$

and for the second-order properties

$$\left\langle \frac{\mathrm{d}^2\mathbf{X}}{\mathrm{d}t^2}, \mathbf{X} \right\rangle = 0. \tag{6.6}$$

We substitute from Eqs. 5.15 and 6.2 in (6.6) and obtain:

$$(\cot \Theta_c - \cot \Theta) \sin \Phi = \frac{\Omega}{P}. \tag{6.7}$$

This is the dual spherical Eular–Savary equation. In analogy with (5.34), the Eular–Savary equation takes the form

$$(\cot \Theta_c - \cot \Theta) \sin \Phi = \Sigma_r, \tag{6.8}$$

where

$$\Sigma_r = \frac{\Omega}{P} = \frac{Q_f}{P} - \frac{Q_m}{P}. \tag{6.9}$$

By separating the real and the dual parts of Eq. 6.8, respectively, we get:

$$(\cot\theta_c - \cot\theta)\sin\varphi = \sigma_r, \tag{6.10}$$

and

$$\varphi^*(\cot\theta_c - \cot\theta)\cos\varphi - \left(\frac{\theta_c^*}{\sin^2\theta_c} - \frac{\theta^*}{\sin^2\theta}\right)\sin\varphi = \sigma_r^*. \tag{6.11}$$

The spherical Eular–Savary equation (6.10) together with (6.11), are called the Disteli formulae of spatial kinematics [28]. Equation 6.9 deals only with the direction of the line **X**, as well as its Disteli axis; Eq. (6.10) is more complicated; by means of (6.11) it may be simplified to

$$\sigma_r\varphi^*\cot\varphi - \left(\frac{\theta_c^*}{\sin^2\theta_c} - \frac{\theta^*}{\sin^2\theta}\right)\sin\varphi = \sigma_r^*. \tag{6.12}$$

If (besides φ^* and φ) θ_c and θ_c^* are known, then θ follows from (6.9), and θ^* from (6.11). Conversely, if we start with θ and θ^*, they determine θ_c and θ_c^*. Hence, during the one-parameter spatial motion H_m/H_f there exists a bio-relationship between an instantaneous fixed line and the Disteli axis.

6.1 A characteristic property of the Disteli axis

Now we give a new characteristic property of the Disteli axis by means of dual angle approximations. Therefore, if we define the dual angle $\Psi = \psi + \varepsilon\psi^* = \Theta_c - \Theta$, where ψ is the angle between the line **X** and its Disteli axis **U** and ψ^* is the minimal distance (along their common perpendicular), then this gives rise to

$$\arccos(\langle \mathbf{X}, \mathbf{U}\rangle) = \Psi. \tag{6.13}$$

This relationship defines a metric for the points on the dual unit sphere. During the motion K_m/K_f (H_m/H_f) this dual angle naturally changes. For the first differential of Ψ with respect to the dual arc length of dS we have from (6.13)

$$\frac{d\Psi}{dS} = \frac{\langle -\mathbf{X}', \mathbf{U}\rangle}{\sqrt{1-(\langle \mathbf{X}, \mathbf{U}\rangle)^2}}. \tag{6.14}$$

Here dash denotes to differentiation w.r.t.s. So, in view of (3.6), we have

$$\frac{d\Psi}{dS} = 0 \Longleftrightarrow \langle -\mathbf{X}', \mathbf{U}\rangle = 0((-K\mathbf{B}) \times \langle \mathbf{X}, \mathbf{U}\rangle = 0) \Longleftrightarrow \mathbf{T}\perp\mathbf{U} \Longleftrightarrow \mathbf{U} = A_1\mathbf{N} + A_2\mathbf{B} \tag{6.15}$$

for some dual numbers $A_1, A_2 \in D$, since $K \neq 0$.

Applying this to Study's map, we obtain as a result the following:

Theorem 6.1 *For the one-parameter motion H_m/H_f, the vanishing of the first differential of the angle and the minimal distance of the directed line* **X** *and its Disteli axis* **U** *is characteristic for the line* **X** *and its Disteli axis* **U** *to lie with the ISA in a hyperbolic line congruence. The common angle and the minimal distance of such lines will be invariant in the first approximation.*

Differentiation of Eq. 6.14 leads similarly to

$$\frac{d^2\Psi}{dS^2} = \frac{(\langle -\mathbf{X}', \mathbf{U}\rangle)'\sqrt{1-(\langle \mathbf{X},\mathbf{U}\rangle)^2} - \langle -\mathbf{X}',\mathbf{U}\rangle(\sqrt{1-(\langle \mathbf{X},\mathbf{U}\rangle)^2})'}{1-(\langle \mathbf{X},\mathbf{U}\rangle)^2}. \tag{6.16}$$

So

$$\frac{d\Psi}{dS} = \frac{d^2\Psi}{dS^2} = 0 \Longleftrightarrow \langle -\mathbf{X}',\mathbf{U}\rangle = (\langle -\mathbf{X}',\mathbf{U}\rangle) = 0,$$

$\Longleftrightarrow$

$$\mathbf{U} = A_1\mathbf{N} + A_2\mathbf{B} \quad \text{and} \quad \langle (A_1\mathbf{N} + A_2\mathbf{B}), K\mathbf{N}\rangle = 0,$$

for some dual numbers $A_1, A_2 \in D$, $\Longleftrightarrow$

$$\mathbf{U} = A_2\mathbf{B},$$

for some $A_2 \in D$, since $K \neq 0$. And $\langle \mathbf{U},\mathbf{U}\rangle = 1 \Rightarrow A_2 = \pm 1$. So

$$\frac{d\Psi}{dS} = \frac{d^2\Psi}{dS^2} = 0 \Longleftrightarrow \mathbf{U} = \pm\,\mathbf{B}. \tag{6.17}$$

So, applying this to Study's map, we have the following theorem

Theorem 6.2 *For the one-parameter motion H_m/H_f, the vanishing of the first and second differential of the angle and the minimal distance of the directed line* **X** *and its Disteli axis* **U** *is characteristic for the correspondence* $\mathbf{B} \to \pm\mathbf{U}$ *for which* **B** *(binormal vector) is the Disteli axis of the trajectory's ruled surface through* **X**. *This correspondence between H_m and H_f leaves the angle and the minimal distance of corresponding lines invariant in the second approximation.*

Differentiation of (6.16) leads similarly to

$$\begin{aligned}\frac{d^3\Psi}{dS^3} &= \frac{1}{1-(\langle \mathbf{X},\mathbf{U}\rangle)^2}\{(\langle -\mathbf{X}',\mathbf{U}\rangle)''\sqrt{1-(\langle \mathbf{X},\mathbf{U}\rangle)^2} - \langle -\mathbf{X}',\mathbf{U}\rangle(\sqrt{1-(\langle \mathbf{X},\mathbf{U}\rangle)^2})'')\} \\ &\quad + \{\langle -\mathbf{X}',\mathbf{U}\rangle)'(\sqrt{1-(\langle \mathbf{X},\mathbf{U}\rangle)^2} - \langle -\mathbf{X}',\mathbf{U}\rangle[1-(\langle \mathbf{X},\mathbf{U}\rangle)^{\mathbf{2}}]'\}(\frac{1}{1-(\langle \mathbf{X},\mathbf{U}\rangle)^2})'.\end{aligned} \tag{6.18}$$

Thus

$$\frac{d\Psi}{dS} = \frac{d^2\Psi}{dS^2} = \frac{d^3\Psi}{dS^3} = 0 \Longleftrightarrow \langle -\mathbf{X}',\mathbf{U}\rangle = \langle -\mathbf{X}'',\mathbf{U}\rangle = \langle \mathbf{X}''',\mathbf{U}\rangle = 0,$$

$\Longleftrightarrow$

$$\mathbf{U} = \pm\,\mathbf{B} \quad \text{and} \quad \langle -\mathbf{U}, (K\mathbf{N})'\rangle = 0,$$

$\Longleftrightarrow$

$$\mathbf{U} = \pm\,\mathbf{B} \quad \text{and} \quad \langle -\mathbf{U}, K'\mathbf{N} + K(-K\mathbf{T} + T\mathbf{B})\rangle = 0,$$

$\Longleftrightarrow$

$$\mathbf{U} = \pm\,\mathbf{B} \quad \text{and} \quad \mp KT = 0 \Longleftrightarrow \mathbf{U} = \pm\,\mathbf{B}, \text{ and } T = 0, \tag{6.19}$$

since $K \neq 0$. Referring to Study's map, we summarize this result in the following theorem:

Theorem 6.3 *For the one-parameter motion H_m/H_f, the vanishing of the first, second and third differential of the angle and the minimal distance of the directed line* $\mathbf{X}$ *and its Disteli axis* $\mathbf{U}$ *characterizes the lines of the torsion line congruence ($T = 0$, is the dual torsion of* $\mathbf{X}$*) for which* $\mathbf{U}$ *is the Disteli axis (ISA of the motion) of the trajectory's ruled surface through* $\mathbf{X}$*. This correspondence between H_m and H_f leaves the angle and the minimal distance of corresponding lines invariant in the third approximation.*

Since $T = 0$, the Disteli axis is fixed to the second order and the generating line of the ruled surface $\mathbf{X} = \mathbf{X}(t)$ moves about it with constant pitch. Thus, locally the ruled surface $\mathbf{X} = \mathbf{X}(t)$ is traced during a helical motion by the line $\mathbf{X}$ located at a distance ψ^* and angle ψ relative to its Disteli axis.

7 Some explanations and an example

First, in this section we demonstrate the use of dual vectors for describing the *ISA* of the one-parameter dual spherical motion K_m/K_f. The one-parameter spatial motion H_m/H_f can be described analytically by the matrix equation

$$\mathbf{x}_f(t) = A(t)\mathbf{x}_m + \mathbf{m}(t), \tag{7.1}$$

where $\mathbf{x}_m, \mathbf{x}_f, \mathbf{m}$ are 3×1 real matrices and $A \in SO(3)$. Here

$$SO(3) = \{A \in O(3) : \det A = 1\}, \quad O(3) = \{A \in \mathbb{R}^3_3 : A^t = A^{-1}\}, \tag{7.2}$$

where A and $\mathbf{m}$ are C^∞ functions of a real parameter t; $\mathbf{x}_m$ and $\mathbf{x}_f$ correspond to the position vector of the same point X, with respect to the orthonormal frames of the moving space H_m and the fixed space H_f, respectively. At the initial time $t = 0$ we consider the case where the orthonormal frames of the moving space H_m and the fixed space H_f are coincident.

The velocity of a fixed point $\mathbf{x}_m \in H_m$ is

$$\mathbf{x}_f^{'} = A^{'}\mathbf{x}_m + \mathbf{m}^{'}, \tag{7.3}$$

since $\mathbf{x}_m$ is fixed. If we replace $\mathbf{x}_m$ in view of (7.1), we get

$$\mathbf{x}_f^{'} = A^{'}A^t\mathbf{x}_f + (\mathbf{m}^{'} - A^{'}A^t\mathbf{m}). \tag{7.4}$$

The matrix $A^{'}A^t$ is 3×3 skew-symmetric as differentiation of $AA^t = I$, where I is the 3×3 unit matrix, gives:

$$A^{'}A^t + AA^{'t} = 0; \quad 0 \text{ is the zero matrix.} \tag{7.5}$$

If we write $\omega = A^{'}A^t$, then (7.5) reduces to

$$\omega + \omega^t = 0. \tag{7.6}$$

Therefore Eq. 7.4 can be rewritten in the form

$$\mathbf{x}_f' = \omega\mathbf{x}_f + (\mathbf{m}' - \omega\mathbf{m}). \tag{7.7}$$

As a direct consequence of this equation, there is a dual vector

$$\mathbf{\Omega}((t) = \boldsymbol{\omega}(t) + \varepsilon\boldsymbol{\omega}^*(t), \tag{7.8}$$

such that

$$\omega\mathbf{x}_f = \boldsymbol{\omega} \times \boldsymbol{x}_f; \quad \vee\mathbf{x}_f \in E^3 \quad \text{and} \quad \boldsymbol{\omega}^* = (\mathbf{m}^{'} - \omega\mathbf{m}). \tag{7.9}$$

For comparison, let us take

$$A(\varphi)=\begin{pmatrix}\cos^2\varphi & \sin\varphi & \sin\varphi\cos\varphi\\ -\sin\varphi\cos\varphi & \cos\varphi & -\sin^2\varphi\\ -\sin\varphi & 0 & \cos\varphi\end{pmatrix},\quad m(\varphi)=\begin{pmatrix}0\\ \mu\cos\varphi\\ \mu\sin\varphi\end{pmatrix};\quad \mu\in\mathbb{R}. \tag{7.10}$$

The application of (7.10) to (7.9) gives

$$\omega=\begin{pmatrix}\sin\varphi\\ \cos\varphi\\ -1\end{pmatrix},\quad \omega^*=\begin{pmatrix}-\mu\cos\varphi(1+\sin\varphi)\\ \mu\sin\varphi(1+\sin\varphi)\\ \mu\cos\varphi(1-\sin\varphi)\end{pmatrix}. \tag{7.11}$$

Therefore the Pfaffian dual vector $\mathbf{\Omega}$ at the instant φ of the one-parameter dual spherical motion K_m/K_f is:

$$\Omega(\varphi)=\omega(\varphi)+\varepsilon\omega^*(\varphi),=\begin{pmatrix}\sin\varphi-\varepsilon\mu\cos\varphi(1+\sin\varphi)\\ \cos\varphi+\varepsilon\mu\sin\varphi(\sin\varphi-1)\\ -1+\varepsilon\mu\cos\varphi(1-\sin\varphi)\end{pmatrix}. \tag{7.12}$$

In view of Theorem (4.2), the fixed axode is given by

$$\mathbf{R}_f(\varphi)=\frac{\mathbf{\Omega}}{\|\mathbf{\Omega}\|}=\frac{1}{\sqrt{2[1-\varepsilon\mu\cos\varphi(1+\sin\varphi)]}}\begin{pmatrix}\sin\varphi-\varepsilon\mu\cos\varphi(1+\sin\varphi)\\ \cos\varphi+\varepsilon\mu\sin\varphi(\sin\varphi-1)\\ -1+\varepsilon\mu\cos\varphi(1-\sin\varphi)\end{pmatrix}. \tag{7.13}$$

This shows that the moving polode on K_m is described by:

$$\Omega_m=\frac{\mathrm{d}M^t}{\mathrm{d}\varphi}M;M=(A+\varepsilon mA), \tag{7.14}$$

where

$$M=\begin{pmatrix}\cos^2\varphi+\varepsilon\sin\varphi\cos\varphi(\sin\varphi-1) & \sin\varphi(1-\varepsilon\mu\cos\varphi) & \sin\varphi\cos\varphi+\varepsilon\mu(\sin^3\varphi+\cos^3\varphi)\\ \sin\varphi\cos\varphi(-1+\varepsilon\mu\cos\varphi) & \cos\varphi+\varepsilon\mu\sin^2\varphi & \sin^2\varphi(-1+\varepsilon\mu\cos\varphi)\\ -\sin\varphi-\varepsilon\mu\cos^3\varphi & -\varepsilon\mu\sin\varphi\cos\varphi & \cos\varphi(1-\varepsilon\mu\sin\varphi\cos\varphi)\end{pmatrix}. \tag{7.15}$$

So that, the moving axode is given by

$$\mathbf{R}_m(\varphi)=\frac{\mathbf{\Omega}_m}{\|\mathbf{\Omega}_m\|}=\frac{1}{\sqrt{2[1-\varepsilon\mu\cos\varphi(1+\sin\varphi)]}}\begin{pmatrix}\sin\varphi+\varepsilon\mu\sin\varphi\cos\varphi(\sin\varphi-1)\\ 1-\varepsilon\mu\sin\varphi\cos\varphi\\ -\cos\varphi+\varepsilon\mu(\cos^2\varphi+\sin^3\varphi)\end{pmatrix}. \tag{7.16}$$

We now calculate the Blaschke invariants of the fixed axode $\mathbf{R_1}=\mathbf{R}_f(\varphi)$. For the motion K_m/K_f, from Eq. 7.13, we can write:

$$\mathbf{\Omega}_f(\varphi)=\Omega\mathbf{R}_1(\varphi);\quad \Omega=\sqrt{2[1-\varepsilon\mu\cos\varphi(1+\sin\varphi)]}. \tag{7.17}$$

Upon differentiating both sides of (7.17), with respect to φ, we have:

$$\frac{\mathrm{d}\mathbf{\Omega}_f}{\mathrm{d}\varphi}=\mathbf{\Omega}_f'=\Omega'\mathbf{R}_1+P\Omega\mathbf{R}_2, \tag{7.18}$$

and

$$\mathbf{\Omega}_f''=(\Omega''-P\Omega)\mathbf{R}_1+(2P\Omega'+P'\Omega)\mathbf{R}_2+PQ_f\Omega\mathbf{R}_3, \tag{7.19}$$

in view of (4.39). Further, let us form the vectorial product

$$\mathbf{\Omega}_f(\varphi)\times\mathbf{\Omega}_f'(\varphi)=P\Omega^2\mathbf{R}_3. \tag{7.20}$$

Now

$$\left\|\mathbf{\Omega}_f(\varphi)\times\mathbf{\Omega}_f^{'}(\varphi)\right\|^2=\langle\mathbf{\Omega}_f(\varphi),\mathbf{\Omega}_f(\varphi),\rangle\langle\mathbf{\Omega}_f^{'}(\varphi),\mathbf{\Omega}_f^{'}(\varphi)\rangle-(\langle\mathbf{\Omega}_f(\varphi),\mathbf{\Omega}_f^{'}(\varphi)\rangle)^2, \tag{7.21}$$

or equivalently

$$\langle\mathbf{\Omega}_f(\varphi),\mathbf{\Omega}_f(\varphi),\rangle\langle\mathbf{\Omega}_f^{'}(\varphi),\mathbf{\Omega}_f^{'}(\varphi)\rangle-(\langle\mathbf{\Omega}_f(\varphi),\mathbf{\Omega}_f^{'}(\varphi)\rangle)^2=P^2\Omega^4. \tag{7.22}$$

Finally, we have:

$$\det(\mathbf{\Omega}_f,\mathbf{\Omega}_f^{'},\mathbf{\Omega}_f^{''})=P^2Q_f\Omega^3. \tag{7.23}$$

On the other hand, from Eq. 7.12, we may obtain that:

$$\Omega_f^{'}(\varphi)=\begin{pmatrix}\cos\varphi-\varepsilon\mu[-\sin\varphi(1+\sin\varphi)+\cos^2\varphi]\\ -\sin\varphi+\varepsilon\mu[\cos\varphi(\sin\varphi-1)+\sin\varphi\cos\varphi]\\ -\varepsilon\mu[\sin\varphi(1-\sin\varphi)]+\cos^2\varphi]\end{pmatrix}, \tag{7.24}$$

and

$$\Omega_f^{''}(\varphi)=\begin{pmatrix}-\sin\varphi-\varepsilon\mu[\cos\varphi(1+\sin\varphi)+3\cos\varphi\sin\varphi]\\ -\cos\varphi+\varepsilon\mu[-\sin\varphi(\sin\varphi-1)+2\cos^2\varphi-\sin^2\varphi]\\ -\varepsilon\mu[\cos\varphi(1-\sin\varphi)-3\cos\varphi\sin\varphi]\end{pmatrix}. \tag{7.25}$$

It follows from Eqs. 7.12 and 7.24 that :

$$\langle\mathbf{\Omega}_f(\varphi),\mathbf{\Omega}_f^{'}(\varphi)\rangle=\varepsilon\mu(\sin^2\varphi-\cos^2\varphi+\sin\varphi) \tag{7.26}$$

So, the result is

$$(\langle\mathbf{\Omega}_f(\varphi),\mathbf{\Omega}_f^{'}(\varphi)\rangle)^2=0. \tag{7.27}$$

Further, we have

$$\langle\mathbf{\Omega}_f^{'}(\varphi),\mathbf{\Omega}_f^{'}(\varphi)\rangle=1+2\varepsilon\mu(-1+2\sin\varphi)\cos\varphi. \tag{7.28}$$

On substituting the values of Ω, $\Omega_f^{'}(\varphi)$ and $\Omega_f^{''}(\varphi)$ in (7.22), we find:

$$1+2\varepsilon\mu(-1+2\sin\varphi)\cos\varphi=\Omega^2P_f^2. \tag{7.29}$$

If we calculate the real and dual parts of Eq. 7.29, we have

$$p=\frac{1}{\sqrt{2}},\quad p^*=\frac{\mu}{2\sqrt{2}}(-1+5\sin\varphi)\cos\varphi. \tag{7.30}$$

We find therefore, by means of (4.37), that the common distribution parameter of the axodes is given by

$$\lambda=\frac{\mu\cos\varphi}{2}(-1+5\sin\varphi). \tag{7.31}$$

In view of (7.12), (7.24) and (7.25), we have

$$\det(\mathbf{\Omega}_f,\mathbf{\Omega}_f^{'},\mathbf{\Omega}_f^{''})=1-3\varepsilon\mu\cos\varphi(1+\sin\varphi). \tag{7.32}$$

We have found, by substituting (7.32) in (7.23), that:

$$1-3\varepsilon\mu\cos\varphi(1+\sin\varphi)=P^2Q_f\Omega^3. \tag{7.33}$$

If we calculate the real and dual parts of this equation, we get

$$q_f=\frac{1}{\sqrt{2}},\quad q_f^*=\frac{-\mu\cos\varphi(1+13\sin\varphi)}{2\sqrt{2}}. \tag{7.34}$$

By Eqs. 7.34 and 4.36 we have:

$$q_m=-\frac{1}{\sqrt{2}},\quad q_m^*=\frac{\mu\cos\varphi(1-11\sin\varphi)}{2\sqrt{2}}. \tag{7.35}$$

By using the real and dual parts of the integral invariants of the axodes, we can find the invariants of the line trajectories associated with the axodes.

8 Conclusion

The starting point of this work is to define the dual version of a curve associated with a curve in the Euclidean 3-dimensional space E^3 given in [26]. We developed this approach in the dual 3-space D^3 to a dual curve associated to a dual curve to provide the requirement of instantaneous kinematics and geometry of spatial motion because the movement of any fixed line in a moving body is associated with the generator of the axode. The kinematics and geometry of ruled surfaces generated in a one-parameter spatial motion have been treated in terms of the invariants of the axodes. According to the derived formulae of the geometry and kinematics of the axodes, one can expect their usefulness in design and analysis of line trajectories associated with the axodes. This study is intended to clarify the subject of second-order one-parameter spatial-motion properties and leads to a general understanding. The results, in addition to their theoretical interest, have applications in the analysis of spatial mechanisms, manipulators and in the prescription of tool paths for manipulator end effectors.

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